values of $1 < \alpha < 1.5$ yield negative Weber numbers, which are obviously meaningless.]

The results of this, as well as Anshus and Goren's (*1*), analysis thus substantiate Yih's (*16*) analysis regarding the controlling effect of surface waves and interfacial velocities on film characteristics.

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NOTATION

c = wave celerity
g = acceleration due to gravity
h = varying film thickness
h_o = average film thickness in wavy flow
m = wave number $(2\pi/\lambda)$
P = pressure
P = capillary pressure
t = time
u = x component of velocity
$\bar{u}$ = average film velocity at film thickness h
U_o = average film velocity at film thickness h_o
v = y-component of velocity
N_{We} = Weber number $(U_o^2\, h_o\, \rho/\sigma)$
N_ζ = dimensionless surface tension parameter, $(\sigma 3^{1/3}/\rho g^{1/3} \nu^{4/3})$
x = cartesian coordinate
y = cartesian coordinate
z = wave forward characteristic $(x - ct)$

Greek Letters

α = dimensionless wave celerity
β = inclination angle of flow to horizontal
δ = dimensionless wave amplitude
λ = wave length
μ = dynamic viscosity
ν = kinematic viscosity
ρ = density
σ = surface tension
ϕ = local deformation of the free surface (see Figure 1)

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Relationships Between Process Equations for Processes in Connection With Newtonian and Non-Newtonian Substances

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Processes in which non-Newtonian substances take part are steadily gaining importance in process-engineering research (*1* to *7*); both the theoretical investigations and the discussion of experimental results being almost exclusively based on the friction law by Ostwald-de Waele. The numerous successes achieved in this connection should not, however, deceive us and we should recognize the limitations of this conception. Relatively simple problems, such as drag in pipelines or stirrer efficiency, are described more or less adequately with this scalar power law. Its insufficiency becomes more apparent as the processes to be investigated become more subtle. Thus, for instance, mixing operations in elastoviscous liquids can only be interpreted by complicated rheological equations of state (*8* to *10*). The endeavor of putting such operations on a better founded theoretical basis is connected with two difficulties. Firstly, the allocation of adequate rheological equations of state to real non-Newtonian substances is possible only in a few cases and secondly, theoretical discussions of such problems usually encounter great mathematical difficulties.

For this reason, the experimental research in process engineering based on similarity methods gains importance. In this connection it would be desirable to find certain universally valid rules for experimental research which are largely independent of the individual properties of the non-Newtonian substances. This includes the following problem: *A process in which a Newtonian liquid takes part can be described by a process equation* $f_{\mathrm{Newt}}(\pi_1, \pi_2, \ldots, \pi_n) = 0$ in which π_i $(i = 1, 2, \ldots, n)$ stands for dimensionless variables due to the problem [pi-variables (*11*)]. The subscript (Newt) of the function symbol indicates that the process takes place in connection with a Newtonian liquid. A corresponding process, with otherwise unchanged conditions, may be carried out in connection with any non-Newtonian substance. It may be described by dimensionless variables π_j^* $(j = 1, 2, \ldots, m)$ and a corresponding process equation $f_{n\text{-Newt}}(\pi_1^*, \pi_2^*, \ldots, \pi_m^*) = 0$. The subscript of the function symbol indicates the non-Newtonian behavior of the substance concerned. Both sets of pi-variables form a complete set of independent quantities $S_{\mathrm{Newt}}(\pi_i)$ and $S_{n\text{-Newt}}(\pi_j^*)$. Each set can be obtained from the respective problem-relevant physical quantities by the well-known methods of dimensional analysis (*11*).

Research in processes with non-Newtonian substances usually starts from the known laws of the corresponding Newtonian case. Therefore, the question of correspondence between the functions f_{Newt} and $f_{n\text{-Newt}}$ as well as the correspondence between π_i and π_j^* is of great importance. This paper deals with this correspondence and continues the discussion of transport processes in non-Newtonian substances found elsewhere (*12*).

A very comprehensive class of non-Newtonian substances whose rheological behavior is described by a relationship between the stress and the rate of strain tensors, as well as their time derivatives and history integrals, was discussed in (*12*). This class of substances comprises also the thixotropic and rheopexic substances whose behavior is determined by previous stress. It has been shown that all the rheological material constants can always be reduced to only two dimensional parameters H and θ and, if necessary, a number of dimensionless parameters π'_k. H has the dimension of viscosity and θ that of time. Between the parameters H and θ and the compliance ψ discussed in (*12*) the relation is $\psi = \theta/H$.

These basic facts lead to a correspondence between the two complete pi-sets S_{Newt} and $S_{n\text{-Newt}}$. The pi-quantities occurring in S_{Newt} can be subdivided into two groups: (a) process variables containing at least one of the quantities characterizing the proper process (for example, equipment dimensions, characteristic speed of rotation, characteristic temperature, as well as process target quantities, such as effective heat-transfer coefficient, period of homogenization, etc.) and (b) substance variables exclusively composed of material constants of all kinds. A complete set of linearly independent pi-quantities can be determined by dimensional analysis (*11*) from the dimension matrix of the process-relevant physical quantities. When changing over from the Newtonian case to the corresponding non-Newtonian case, the column concerning the viscosity η in the dimension matrix is replaced by the equivalent H column. In addition there is also the θ column. If the transition (Newt) $\rightarrow$ (n-Newt) causes further nonrheological quantities, this leads to further process variables [see example (a)]. The material parameters π'_k need not appear in the matrix. It is shown (*12*) that the rank of the dimension matrix is not increased. Therefore, regardless of the rheological properties of the non-Newtonian substances, there are the following consequences for $S_{n\text{-Newt}}$:

1. All pi-variables occurring in S_{Newt} also occur in $S_{n\text{-Newt}}$, the viscosity η being replaced by H.
2. In the group of the process variables of $S_{n\text{-Newt}}$ there occurs at the most one additional pi-variable containing θ.
3. Furthermore, the complete set of material parameters π'_k appears in the group of the substance variables of $S_{n\text{-Newt}}$.

All the characteristic rheological parameters, which are not completely known in practice, are thus contained, except for H and θ, only in the group of the substance variables of $S_{n\text{-Newt}}$. This fact is very important for experimental research on processes with non-Newtonian substances. No matter how complicated the behavior of the non-Newtonian substance to be investigated is (the rheological equation of state need not even be known), there is only one parameter more than in the Newtonian case, if the experimental investigation is restricted to a single substance only.

We have so far assumed that the substance parameters are independent of the temperature. The correspondence relation is, however, maintained if rheological and nonrheological substance parameters depend on temperature and the temperature coefficients of a first or higher order belong to the process-relevant quantities. In this case we get additional pi-quantities. It is always possible to take account of the new variables by adding only a single parameter $1/\eta \cdot \partial\eta/\partial T \cdot \Delta T$ to the group of the process variables of S_{Newt} or respectively adding $1/H \cdot \partial H/\partial T \cdot \Delta T$ to the group of the process variables of $S_{n\text{-Newt}}$. ΔT is a characteristic temperature difference in the process. All other temperature coefficients, regardless of order, can be included in the group of substance variables in the form of quotients.

In conclusion, a short remark on the friction law according to Ostwald-de Waele. This power law in the form $\tau = K \cdot D^m$ is inconsistent with dimensional analysis because the substance parameter $K = \theta^{m-1} \cdot H$ is a composite quantity whose physical dimension depends on m. Hence, it has been suggested (*12*) to put this law for dimensional investigations in the form $\tau \cdot \psi = (\theta \cdot D)^m$ or $\tau \cdot \theta/H = (\theta \cdot D)^m$, respectively. This fact is well known, but is not sufficiently regarded in the literature. Consequently, process equations are obtained whose variables form an incomplete pi-set. Such relations are therefore possible but not *a priori* correct from the point of view of dimensional analysis. It is, however, important that when the power law, which constitutes only an approximation, is applied in this manner, the basic facts discussed here are covered up. For more detailed information see reference *12*.

The following is a brief discussion of two examples concerning the correspondence explained:

1. The power P for stirring without a vortex is described in the Newtonian case by a relation between the two dimensionless variables (*13* to *15*)

$$f_{\mathrm{Newt}}\left(\frac{\mathrm{P}}{\rho n^3 d^5}, \frac{\rho n d^2}{\eta}\right) = 0 \qquad (1)$$

The function f_{Newt} depends only on the geometry of the agitator and the vessel. In accordance with the correspondence relation, the power characteristic in the case of a non-Newtonian substance is given by a relation between three dimensionless variables

$$f_{n\text{-Newt}}\left(\frac{\mathrm{P}}{\rho n^3 d^5}, \frac{\rho n d^2}{H}, n\theta\right) = 0 \qquad (2)$$

The function $f_{n\text{-Newt}}$ depends now on both the geometrical conditions and the rheological properties of the material. If the material exhibits a pronounced Weissenberg effect (*16*) in which the material winds upwards on the agitator shaft, the acceleration of gravity may also need to be considered as a process-relevant quantity. This results in the addition of a further pi-variable, such as $n^2 d/g$, to Equation (2).

2. As a second example, the effective heat transfer coefficient h of stationary heat transfer between a rotating sphere and the unlimited medium surrounding it will be discussed. Both the power dissipation in the fluid and the influence of the temperature dependence of the substance parameters of the fluid on the process are considered. Let the temperature dependence of the material coefficients be represented by the logarithmic temperature coefficients $1/\eta_o \cdot (\partial\eta/\partial T)_o$, $1/\rho_o \cdot (\partial\rho/\partial T)_o$, etc., the subscript o referring to T_o, the temperature of the fluid at infinity.

In the Newtonian case, as can easily be verified by dimensional analysis, this process can be described by a relation between $\mathrm{N}_{\mathrm{Nu}} \equiv hd/k_o$, $N_{Re} \equiv \rho_o n d^2/\eta_o$, $N_{Br} \equiv \eta_o n^2 d^2/(k_o \cdot \Delta T)^{(1F)}$, $N_{Pr} \equiv \eta_o c_o/k_o$, $(\partial\eta/\partial T)_o \cdot \Delta T/\eta_o$ and the corresponding quotients $(\eta_o/\rho_o) \cdot (\partial\rho/\partial T)_o/(\partial\eta/\partial T)_o$, $(\eta_o/k_o) \cdot (\partial k/\partial T)_o/(\partial\eta/T)_o$ and $(\eta_o/c_o) \cdot (\partial c/\partial T)_o/(\partial\eta/\partial T)_o$. Here ΔT is the difference between the constant temperature of the rotating sphere and T_o. If the problem is restricted to a given fluid, the substance variables N_{Pr} and the quotient parameters will be fixed and the problem reduced to the relation

$$f_{\text{Newt}} \left[N_{Nu}, N_{Re}, N_{Br}, \left(\frac{\partial \eta}{\partial T} \right)_o \cdot \Delta T / \eta_o \right] = 0 \quad (3)$$

Experiments on elastoviscous liquids have shown (*8* to *10*) that the corresponding process in a non-Newtonian medium is coupled with a considerably more complicated flow pattern. However, the heat transport in this case is given by the following relation:

$$f_{n\text{-Newt}} \left[\frac{hd}{K_o}, \frac{\rho_o n d^2}{H_o}, \frac{H_o n^2 d^2}{k_o \cdot \Delta T}, \left(\frac{\partial H}{\partial T} \right)_o \cdot \Delta T / H_o, \; n\theta_o \right] = 0$$

which in comparison with the Newtonian case shows only one additional variable. The function $f_{n\text{-Newt}}$ is determined by the rheological behavior of the medium.

NOTATION

c = specific heat
d = characteristic length
D = rate of shear
f = function
h = heat transfer coefficient
k = thermal conductivity
K = rheological parameter
m = rheological parameter
n = number of revolutions
N_{Br} = Brinkman number
N_{Nu} = Nusselt number
N_{Pr} = Prandtl number
N_{Re} = Reynolds number
P = power
S = symbol for set of pi-variables
T = temperature

Greek Letters

η = viscosity
H = rheological parameter
θ = rheological parameter
π_i = dimensionless variables
ρ = density
τ = shear stress
ψ = compliance

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Laminar Flow of Two Immiscible Liquid Falling Films

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The problem of steady state laminar gravity flow over a flat surface of two immiscible liquids involves the simultaneous solution of the equations of motion for each liquid. Consider the region to be sufficiently far from the ends of the wall that the entrance and exit disturbances are negligible. Denoting the liquid which adheres to the wall as 1 and the outer liquid as 2, the equations of motion and boundary conditions for Newtonian fluids are

$$\mu_i \frac{d^2 v_i}{dx^2} = -\rho_i g_y \quad (1)$$

$$x = 0, \quad v_1 = 0 \quad (2)$$

$$x = \delta_1, \quad v_1 = v_2 \quad (3)$$

$$x = \delta_1, \quad m \frac{dv_1}{dx} = \frac{dv_2}{dx} \quad (4)$$

$$x = \delta_1 + \delta_2 = \delta_T, \quad \frac{dv_2}{dx} = 0 \quad (5)$$

The solution to Equation (1) is a parabola for each liquid. Introducing the boundary conditions stated in Equations (2) to (5) gives the local velocities

$$v_1 = \frac{\rho_1 g_y}{2\mu_1} \left[\left(\frac{2\delta_2}{r} + 2\delta_1 \right) x - x^2 \right] \quad (6)$$

for $0 \leq x \leq \delta_1$, and

$$v_2 = \frac{\rho_2 g_y}{2\mu_2} \left[\left(\frac{r}{m} - 1 \right) \delta_1^2 + 2\delta_1\delta_2 \left(\frac{1}{m} - 1 \right) + 2\delta_T \, x - x^2 \right] \quad (7)$$

for $\delta_1 \leq x \leq \delta_T$.

The average velocity in each film is given by

$$\bar{v}_i = \int v_i dx / \int dx \quad (8)$$

where the integration is over the respective film thickness. Substituting Equations (6) and (7) into Equation (8) and integrating yields

$$\bar{v}_i = \frac{\rho_i g_y \delta_i^2}{3\mu_i} G_i \quad (9)$$

where

$$G_1 = 1 + 3/(2 \; r\varphi) \quad (10)$$